S. Ma'u – 29/5/08 – Gluing quilted disks

- A <u>non-unital A_∞-category</u>:

  $\mathbb{K}$ field, $\quad$ $\mathrm{Ob}\,\mathcal{A}$ objects

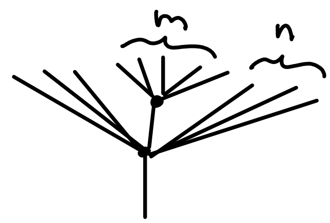
  $$\begin{array}{c}\mathrm{Ob}\,\mathcal{A}\\ \cup \\ X_0, X_1 \longmapsto \hom_{\mathcal{A}}(X_0, X_1) \text{ graded } \mathbb{K}\text{-vect. spaces}\end{array}$$

  $$d \geq 1 \leadsto \mu^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{A}}(X_0, X_d)[2-d]$$

  <u>A_∞-associativity relations</u>:

  $$\forall d \geq 1, \quad \sum_{m,n} (-1)^{\maltese} \mu^{d-1-m}(a_d, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n \dots a_1) = 0$$

  pictorially:



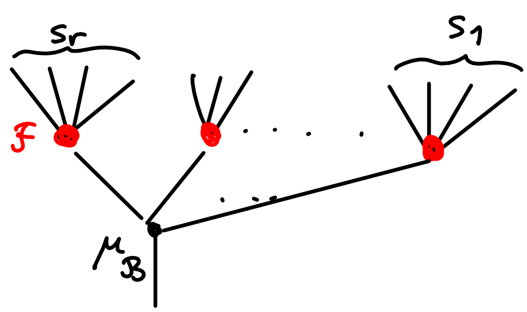
- <u>non-unital A_∞-functor</u> $\mathcal{F} : \mathcal{A} \to \mathcal{B}$:

  - $\mathcal{F} : \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$
  - $\forall d \geq 1$, $\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \dots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}(X_0), \mathcal{F}(X_d))[1-d]$

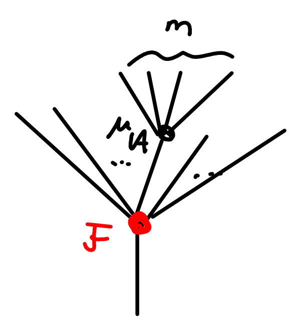
  $$\text{s.t.} \quad \sum \sum \mu^r_{\mathcal{B}}\left(\mathcal{F}^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1)\right)$$

  $$= \sum_{m,n} (-1)^{*} \mathcal{F}^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \dots, a_{n+1}), a_n \dots a_1)$$

Pictorially:



and



I) <u>The enlarged Fukaya category $Fuk^{*}(M)$</u>:

(M, ω) symplectic $(+ \dots)$

- <u>Objects</u> = <u>gen^{d} Lagrangian submanifolds</u> $\underline{L} = * \xrightarrow{L_{-n}} M_{-n} \to \dots \xrightarrow{L_{-2,-1}} M_{-1} \xrightarrow{L_{-1,0}} M_0 = M$

• given $\underline{L}^0, \underline{L}^1$: $\hom_{Fuk^{\#}M}(\underline{L}^0, \underline{L}^1) = CF(\underline{L}^0, \underline{L}^1)$

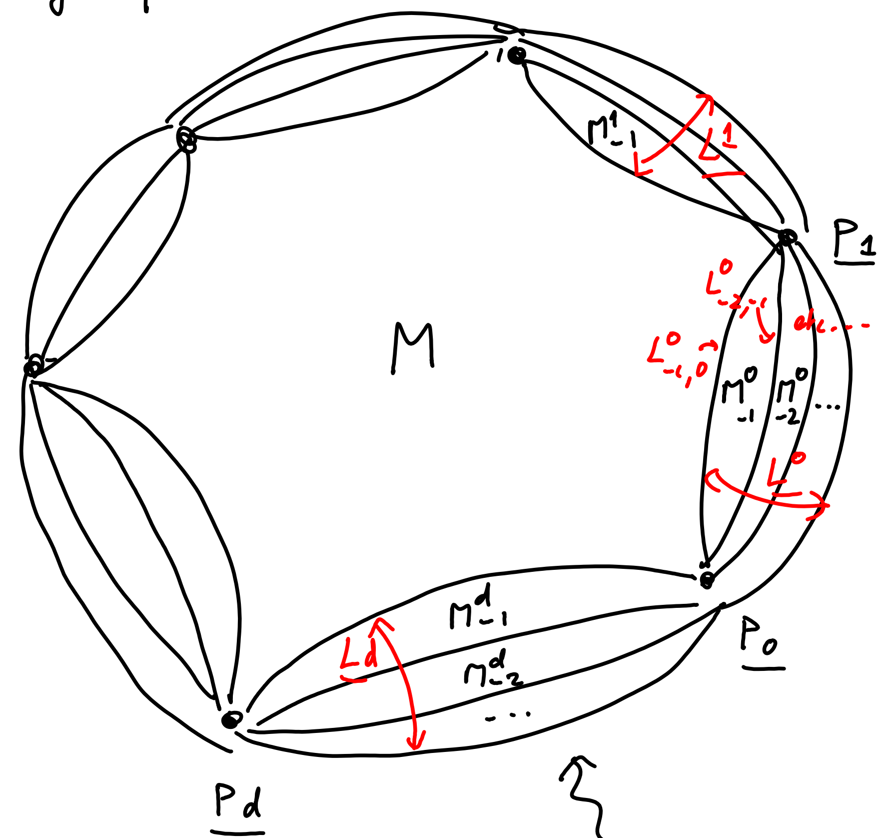
$$= \bigoplus_{p \in \mathcal{I}(\underline{L}^0, \underline{L}^1)} \mathbb{Z}_2 \langle p \rangle$$

generalized intersections (= intersection in big product)

• Compositions: $\mu^d : CF(\underline{L}^{d-1}, \underline{L}^d) \otimes \dots \otimes CF(\underline{L}^0, \underline{L}^1) \longrightarrow CF(\underline{L}^0, \underline{L}^d)$

$$(p_d, \dots, p_1) \longmapsto \sum_{p_0 \in \mathcal{I}(\underline{L}^0, \underline{L}^d)} \left| \mathcal{M}(p_d \dots p_0)^{\overset{\text{0-dim'l part}}{0}} \right| \langle p_0 \rangle$$

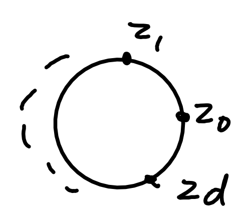
moduli space of quilted disks



all these side strips have a fixed width $\delta = 1$
(A_∞-structure depends on choice of δ...)

Domains for these quilted discs are parametrized by

$$\mathcal{R}^{d+1} = Conf^{d+1}(\partial D) / Aut(D)$$



Compactifies to $\overline{R}^{d+1}$ Deligne-Mumford-Stasheff associahedra

$\uparrow$ $(d-2)$-dim'l mfld with corners (convex polytope)

II) <u>Functors associated to Lagr. correspondences:</u>

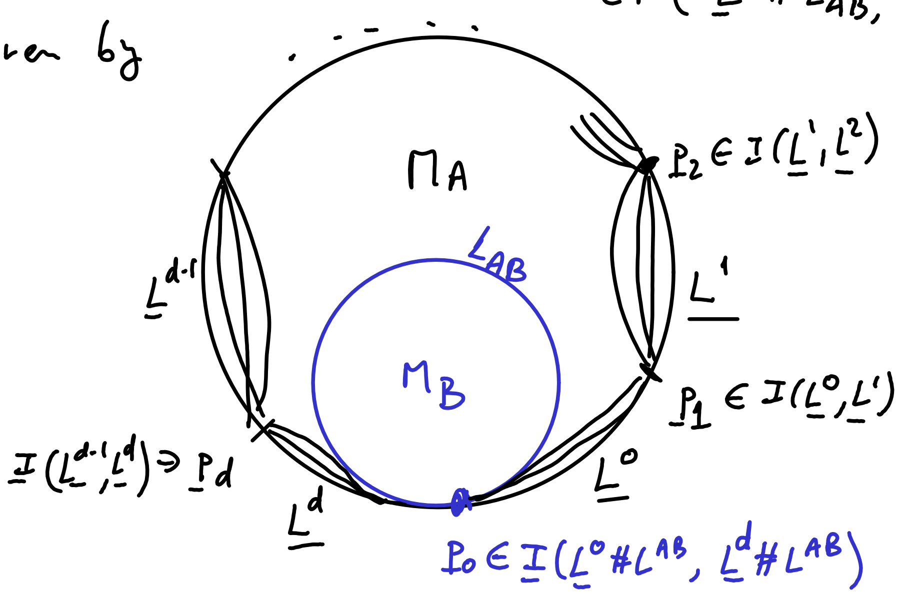
$$(M_A, \omega_A) \xrightarrow[L_{AB} \subset M_A^- \times M_B]{} (M_B, \omega_B) \rightsquigarrow A_\infty\text{-functor } \mathcal{F}_{AB} : \mathrm{Fuk}^\#(M_A) \to \mathrm{Fuk}^\#(M_B)$$

On <u>objects</u>: $\underline{L} \mapsto \underline{L} \# L_{AB} =: \underline{L}_{AB}$

ie. $$\overbrace{\underbrace{* \to M_{-n} \to \dots \to M_A}_{\in \mathrm{Ob}\, \mathrm{Fuk}^\#(M_A)} \xrightarrow{L_{AB}} M_B}^{\in \mathrm{Ob}\, \mathrm{Fuk}^\#(M_B)}$$

On <u>morphisms</u>: $$\mathcal{F}^d : CF(\underline{L}^{d-1}, \underline{L}^d) \otimes \dots \otimes CF(\underline{L}^0, \underline{L}^1)$$
$$\downarrow$$
$$CF(\underline{L}^0 \# L_{AB}, \underline{L}^d \# L_{AB})$$

given by



quilted disks

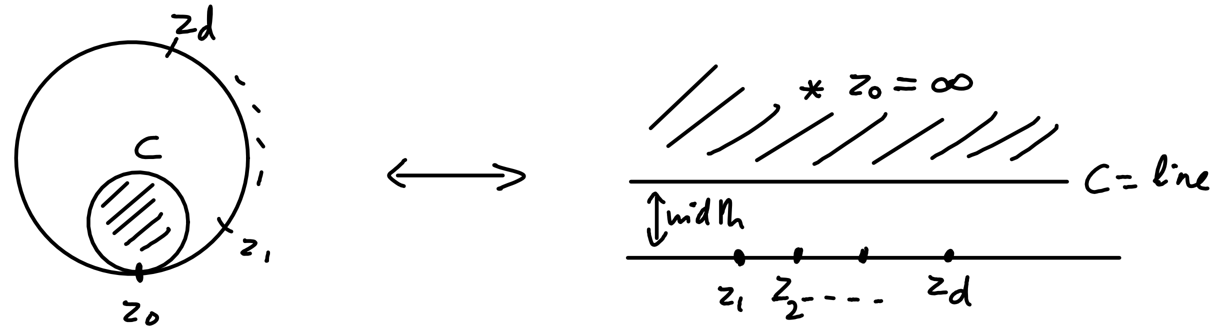
Domains of these are parametrized by quilted disks moduli QD_{d+1}

moduli space: $QD_{d+1} \ni (\underset{\uparrow \atop \text{disk}}{D}, \underset{\uparrow \atop \text{inner circle}}{C}, \underbrace{z_0, \dots, z_d})$ ← boundary marked pts

$C \subset D$ $\qquad \in \mathrm{Conf}_{d+1}(\partial D)$

$C \cap \partial D = \{z_0\}$

Note:

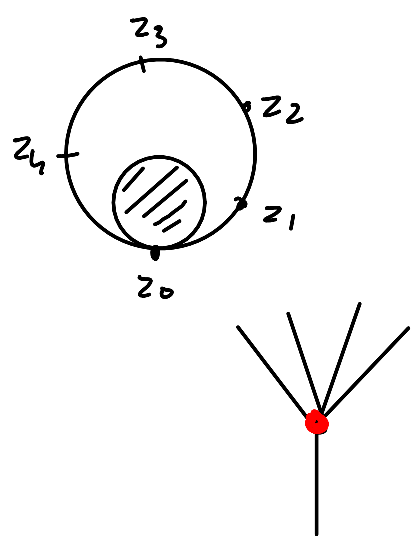


up to translation and dilation

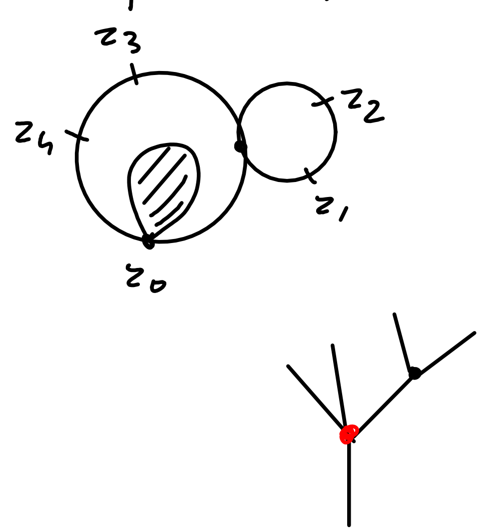
$$\mathcal{R}_1^{d+1} = QD_{d+1} / \mathrm{Aut}\, D \quad \text{has dim.} = d-1.$$

- Compactification $\overline{\mathcal{R}_1^{d+1}}$ stratified by combinatorial type

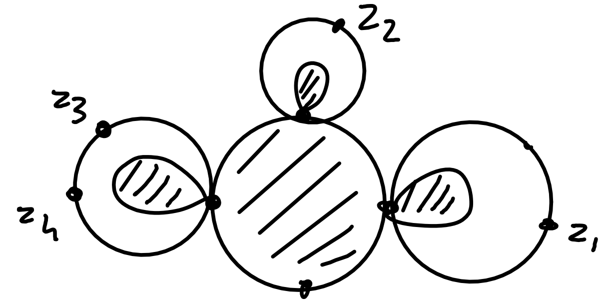
$\longleftrightarrow$ bicolored trees: (stable)
- → vertices = disk components
- → finite edges = nodes
- → semi ∞ edges = markings, w/ root z_0
- → colored vertices = quilted components



compactifies e.g. to

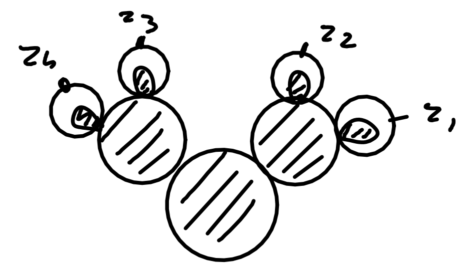


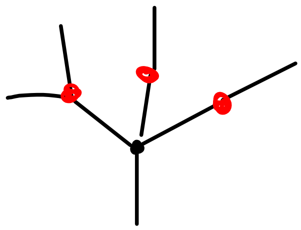
or

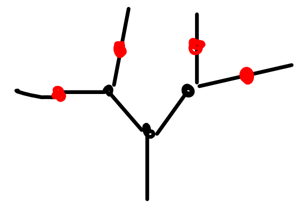


,

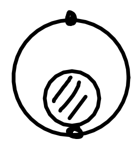
deepest stratum (stable)

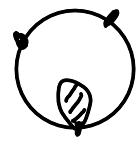


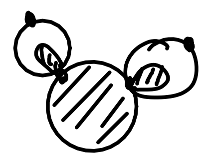
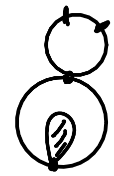


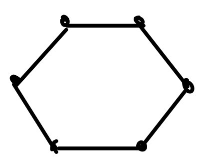


<u>Pictures:</u>

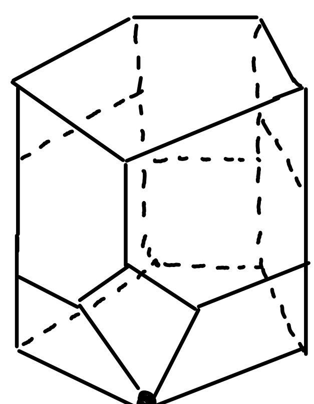
$d = 1$  $=$ $\{$  $\}$

$d = 2$  (main stratum 

ends $=$  and )

$d = 3$:  hexagon

$d = 4$:



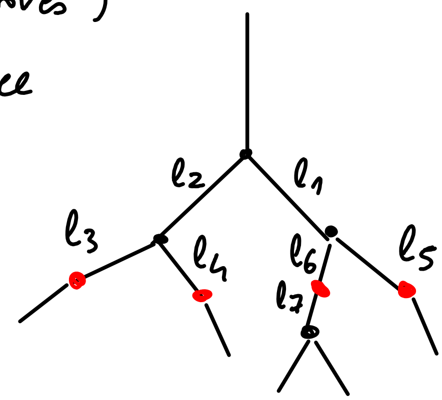
this point = "siamese Mickey Mouse" above

non simplicial

(one vertex has 4 faces adjacent to it)

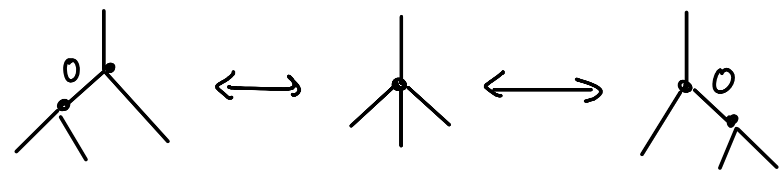
- <u>Metric bicolored trees</u> ($\leftrightarrow$ dual decomposition)
  (a.k.a. "mangroves")

  Γ bicolored tree



$l_i \in [0, \infty)$

with equiv^ce relation



collapsing of length 0 edges

& <u>relations</u>: distances from root vertex to colored vertices are all equal.

(in example: $l_3 = l_4$, $l_5 = l_6$, $l_2 + l_3 = l_1 + l_5$)

$$\overline{MBT_{d+1}} = \bigcup_{\Gamma} M\Gamma / \sim$$

combinatorial type of tree Γ

<u>Prop</u>: $\overline{MBT_{d+1}} \cong \overline{\mathcal{R}}_1^{d+1} \cong J_d$ multiplihedra.

metric bicolored trees — moduli of stable quilted discs.

<u>NB</u>: these guys are polytopes, $\leftrightarrow$ toric varieties
(not smooth: cf. $\overline{\mathcal{R}}_1^4$ non-simplicial)

<u>Rmk</u>: the DM moduli space is birational to a toric variety
$\rightsquigarrow$ $\overline{\mathcal{R}}_1^{d+1}$ = real positive part ....